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An asymptotic approach in problems of crack identification☆

A.O. Vatul'yan, O.V. Yavruyan

Rostov-on-Don, Russia

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Abstract

An asymptotic approach to solving problems of the identification of a rectilinear crack of small relative size is presented. The solution of the direct problem is reduced to solving a boundary integral equation. Using the proposed approach, its kernel is investigated, and the main part of the asymptotic form is singled out. The inverse problem of determining the crack parameters from prescribed information on the amplitudes of the displacement on the boundary of a layer is solved. Transcendental equations are obtained, from which the characteristics of a crack are determined in stages. Numerical results of the solution of the inverse problem are presented.

Problems of identifying internal defects located at the joint of regions, arising as a result of poor bonding (or welding) of materials,[1,2] vertical cracks in a layer or half-space and rectilinear defects emerging at the boundary of a region[3] have been investigated in some detail. A review is available on the current state of research on the theory of inverse crack problems.[2] Problems of identifying cracks in finite bodies have been investigated far less; the greatest progress in solving such problems has been possible when *a priori* information has been available as to the particular plane in which the crack, or system of cracks, is located. In this case, the identification problem can be divided into the problem of determining the parameters of the plane to which the crack belongs, its centre in this plane and its characteristic linear dimensions. The determination of the plane involves introducing a certain "non-reciprocity" functional, with the aid of which it is possible to single out the "main" parameters and to find them from certain simple relations.[2] At the same time, problems of identifying internal crack-like defects of arbitrary configuration have not been investigated to any great extent, although direct problems concerning the construction of displacement fields in bodies with such defects have been studied in some detail.[4] This is due to the increase in the number of crack-determining parameters, which leads to an increase in the dimensions of the search space. Here, *a priori* information on the size of cracks can considerably simplify the procedure for crack identification.

When investigating the multiparameter problem of the vibrations of an elastic layer of thickness h with a crack of characteristic length l, we will distinguish three dimensionless parameters $\varepsilon_1 = l/h$, $\varepsilon_2 = \omega h/c$ (c is the characteristic velocity of waves in the medium) and $\varepsilon_3 = \omega l/c = \varepsilon_1 \varepsilon_2$ (ω is the frequency of the vibrations). Note that the solution of the inverse identification problem is constructed in the region of variation of the parameter $\varepsilon_2 \geq \varepsilon_*$, which corresponds

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E-mail address: vatulyan@aaanet.ru (A.O. Vatul'yan).

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to considering vibrations at a frequency higher than the critical frequency when there are travelling waves in the layer.

Below, an asymptotic approach is proposed that can be examined in the range of variation of the parameters $\varepsilon_1 \ll 1$, $\varepsilon_2 \geq \varepsilon_*$, which corresponds to the case of cracks of small relative size, and here the procedure for solving both the direct and the inverse problem can be simplified considerably. It must also be pointed out that recommendations of practical flaw detection on choosing the vibration frequency correspond to values of the dimensionless parameter ε_3 of the order of unity and higher. As shown by calculations to reconstruct the crack parameters, the asymptotic approach also covers the range of variation of the parameters when the wavelength of the probed signal is greater than the crack length, i.e. $\varepsilon_3 \leq 1$.

1. Statement of the problem

The problem of identifying an internal crack-like defect in an orthotropic layer of thickness h from the displacement fields on part of the boundary of the layer S_{21} is considered. Vibrations are caused by a load applied to part of the upper boundary of the layer S_{20}. The lower boundary of the layer S_1 is rigidly clamped. The crack is modelled as a mathematical cut with sides $s_0^{\pm}$ on which the components of the displacement field suffer a discontinuity

$$\chi_i = u_i\big|_{S_0^+} - u_i\big|_{S_0^-}$$

which is characterized by the components of the vector function of the opening of the crack. On the basis of dislocation theory,[5] the action of the crack is replaced by the action of fictive mass forces, which are expressed in terms of the components of the opening of the crack

$$f_i = -[c_{ijkl} n_k^+ \chi_l \delta(\zeta)]_{,j}$$

where $n_j^{\pm}$ are the components of the unit vectors of the normal to the sides of the crack $s_0^{\pm}$.

The steady-state vibration is considered, which enables us to separate out the time factor and to represent components of the displacement vector in the form $u_j = u_j(x)e^{-i\omega t}$, where $x = (x_1, x_3)$ and ω is the frequency of the vibrations. Then, after separating out the time factor, the problem is described by the following boundary-value problem

$$\sigma_{ij,j} + \rho\omega^2 u_i + f_i = 0, \quad \sigma_{ij} = C_{ijkl} u_{k,l} \tag{1.1}$$

$$u_i\big|_{S_1} = 0, \quad \sigma_{i3}\big|_{S_{20}} = p_i, \quad \sigma_{i3} = 0, \quad x \notin S_{20} \tag{1.2}$$

$$\sigma_{ij} n_j^{\pm}\big|_{S_0^{\pm}} = 0, \quad i, j = 1, 2, 3 \tag{1.3}$$

where ρ is the density of the medium and C_{ijkl} are the components of the constants of elasticity tensor of the material, that satisfy the usual relations of symmetry and positive definiteness.

The problem of crack identification using is solved information on the displacement field, measured on a part of the upper boundary of the layer $S_{21} = \{x_1 \in [c, d], x_3 = h\}$

$$u_i\big|_{S_{21}} = u_i^*(x_1), \quad x_1 \in S_{21} \tag{1.4}$$

Since the region considered contains an infinitely distant point, the formulation of the problem of the condition of the radiation of waves at infinity, in the formulation of which the limiting absorption principle is used,[6] is closed.

We will select the coordinate axes such that the axes of elastic symmetry of the orthotropic material coincide with the axes of the coordinate system, and we will then assume that the crack is a tunnel slit whose axis coincides with the x_2 axis. Then, the initial problem (1.1)–(1.3) breaks down into two secondary problems: the problem of anti planar vibrations of an orthotropic layer with a crack of arbitrary configuration (Problem 1), when the component $u_2 = u(x_1, x_3)$ is non-zero and in problem (1.1)–(1.3) we assume $i, j = 2$, and the plane problem of the vibrations of a layer with a crack (Problem 2); in this case, the components $u_1(x_1, x_3)$, $u_3(x_1, x_3)$ will be non-zero, and in problem (1.1)–(1.3) we assume that $i, j = 1.3$.

Below we will consider the case of a concentrated load of magnitude p_0, applied at a point with coordinates $(-L, h)$ $(L>0)$, in the tangential direction to the boundary for the antiplanar problem and in the normal direction for the plane problem.

2. The solution of the direct problems

2.1. Reduction to integral equations

Solutions of the direct problems are constructed using Green's functions for the layer[4,7,8] and the reciprocity theorem.[5]

In the case of anti planar vibrations (Problem 1), we have the following representation of the displacement field in the layer

$$
\begin{aligned}
&u(\xi) = u^s(\xi) + \int\limits_l k(\xi, x)\chi(x)dl_x,\\
&u^s(\xi) = -\frac{p_0}{2\pi}\int\limits_\sigma \frac{\mathrm{sh}(\lambda\xi_3)}{\lambda\,\mathrm{ch}(\lambda h)} e^{-i\alpha_1(L+\xi_1)}d\alpha_1, \quad \xi = (\xi_1, \xi_2)\in S\\
&k(\xi, x) = \frac{1}{2\pi}\int\limits_\sigma \left[\frac{g_2^+ sh(\lambda\xi_3)e^{-\lambda(h-x_3)} - g_2^- ch(\lambda(h-\xi_3))e^{-\lambda x_3}}{\lambda ch\lambda h} + \frac{g_1}{\lambda}e^{-\lambda|x_3-\xi_3|}\right]e^{i\alpha_1(x_1-\xi_1)}d\alpha_1\\
&g_1 = i\nu n_1(x)\alpha_1 - \mathrm{sign}(x_3-\xi_3)\lambda n_3(x), \quad g_2^{\pm} = i\nu n_1(x)\alpha_1 \pm \lambda n_3(x)\\
&\lambda^2 = \nu\alpha_1^2 - k^2, \quad \nu = C_{66}/C_{44}, \quad k^2 = \rho\omega^2/C_{44}
\end{aligned}
\tag{2.1}
$$

Here σ is the contour in the complex plane, which is selected in accordance with the limiting absorption principle and gets round the singularities of the integrands in a certain way.[8]

For plane strain (Problem 2) the components of the displacement field are defined by similar representations

$$
u_m(\xi) = u_m^s(\xi) + \int\limits_l \sigma_{ij}^{(m)}(x, \xi)n_i\chi_j dl_x, \quad i, j, m = 1, 3, \quad \xi\in S
\tag{2.2}
$$

where $\sigma_{ij}^{(m)}$ are the components of the stress tensor (singular solutions), determined using representations for Green's functions and Hooke's law. In the case of a region of the layer type, similar to the well-known procedure,[5] Green's functions can be represented in the form of Fourier integrals along the contour σ similar to representation (2.1). In expressions (2.1) and (2.2), the first terms are standard fields comprising the displacement fields in a medium without a defect, and the second terms are governed by the presence of a crack in the layer.

One of the most effective methods for determining jumps in displacements on a crack is to construct systems of boundary integral equations (BIEs),[9,10] which are formulated on the basis of representations (2.1), (2.2), taking into account the boundary conditions on the crack (1.3). In the case of Problem 1, we have one BIE

$$
\begin{aligned}
&\int\limits_l K(x, y)\chi(x)dl_x = F(y), \quad y\in l,\\
&K(x, y) = \frac{1}{2\pi}\int\limits_\sigma R(\alpha_1, x, y)e^{i\alpha_1(x_1-y_1)}d\alpha_1\\
&F(y) = -\frac{p_0}{2\pi}\int\limits_\sigma \frac{\nu i\alpha_1 n_1(y)\mathrm{sh}(\lambda y_3) - n_3(y)\lambda\mathrm{ch}(\lambda y_3)}{\lambda\mathrm{ch}(\lambda h)}e^{-i\alpha_1(L+y_1)}d\alpha_1
\end{aligned}
\tag{2.3}
$$

In the case of Problem 2, we obtain a system of two BIEs

$$\int_l K_{ji}(x,y)\chi_i(x)dl_x = F_j(y), \quad y \in l$$
$$K_{ji}(x,y) = C_{jrms}\frac{\partial}{\partial y_s}\sigma_{ki}^{(m)}(x,y)n_k(x)n_r(y) = \frac{1}{2\pi}\int_\sigma e^{i\alpha_1(x_1-y_1)}k_{ji}(\alpha_1,x,y)d\alpha_1 \qquad (2.4)$$
$$F_i(y) = -C_{ijms}\frac{\partial}{\partial y_s}\overset{s}{u}_m(y)n_j(y), \quad j,i,k,m,s = 1,3$$

BIEs (2.3), (2.4) in the general case of curved cracks can be solved numerically using the boundary element method.[6,7] After finding the functions of the opening of the crack using formulae (2.1), (2.2), it is possible to calculate the field on the surface of the layer.[7,8]

In a series of calculations, we investigated the dependence of the displacement fields on the surface of the layer on the parameters of the rectilinear crack: its depth, length, and angle of inclination to the lower boundary of the layer.

The procedure for solving BIEs is fairly complex and requires considerable computational work. Further simplification of the BIEs is possible by means of an asymptotic analysis of the problem for small relative dimensions of the defect.

3. An asymptotic approach to calculating the wave fields

We will consider in more detail the BIE for Problem 1, having represented the kernel of the integral operator in the form of the sum of regular and irregular parts

$$\int_l [K_0(x,y) + K_1(x,y)]\chi(x)dl_x = F(y), \quad y \in l$$

$$K_s(x,y) = \frac{1}{2\pi}\int_\sigma R_s(\alpha_1,x,y)e^{i\alpha_1(x_1-y_1)}d\alpha_1, \quad s = 0,1$$

An investigation of the asymptotic forms of integrands $R_0(x, y)$ and $R_1(x, y)$ as $|\alpha_1| \to \infty$ also revealed[8] that $R_1(\alpha_1, x, y) \to 0$ for all $x, y \in l$, while when $x_3 = y_3$ the function $R_0(\alpha_1, x, y)$ is an increasing function as $|\alpha_1| \to \infty$, and the corresponding integrals are understood in the sense of the finite Hadamard value.[11]

For cracks allowing of parameterization

$$x_j = q_j(t), \quad y_j = q_j(\tau), \quad t,\tau \in [-1,1]$$

$$q_1(t), q_3(t) \in C^1[-1,1], \quad g(t) = \sqrt{q_1'^2(t) + q_3'^2(t)} \neq 0$$

BIE (2.3) can be reduced to the form

$$\int_{-1}^{1}\left[\frac{G(t,\tau)}{(t-\tau)^2} + \tilde{K}_1(t,\tau)\right]\chi(t)g(t)dt = \tilde{F}(\tau)$$
$$G(t,\tau) = \frac{\sqrt{\nu}[\nu q_3'(\tau)q_3'(t) + q_1'(t)q_1'(\tau)]}{g(t)g(\tau)(q_1'^2(\tau) + \nu q_3'^2(\tau))}, \quad \tilde{K}_1(t,\tau) = K_1(x(t),y(\tau)), \quad \tilde{F}(\tau) = F(y(\tau)) \qquad (3.1)$$

and here the kernel $\tilde{K}_1(t, \tau)$ is a continuous function.

Let us consider in more detail the case of a rectilinear crack of length $2l_0$ with an angle of inclination θ to the lower boundary of the layer, the mid-point of which lies on the Ox_3 axis at a distance d_0 from the lower boundary. The quantity L is the distance from the point of application of the load along the Ox_3 axis. The parametric equations of such a crack have the form

$$q_1(t) = l_0 t\cos\theta, \quad q_3(t) = d_0 + l_0 t\sin\theta, \quad t\in[-1,1] \tag{3.2}$$

Taking into account the parameterization introduced, and evaluating the integral on the right-hand side using the theory of residues, Eq. (3.1) becomes

$$l_0^{-1}\int_{-1}^{1}\left[\frac{\sqrt{\nu}}{(t-\tau)^2}+l_0^2\tilde{K}_1(t,\tau)\right]\tilde{\chi}(t)dt = \tilde{F}(\tau), \quad \tau\in[-1,1]$$

$$\begin{aligned}\tilde{F}(\tau) = {}&\frac{p_0}{h}\sum_{n=1}^{\infty}(-1)^{n+1}(-\operatorname{sign}(L+l_0\tau\cos\theta)\sin\theta\sin(\lambda_n^0(d_0+l_0\tau\sin\theta))-\\&-\frac{i\lambda_n^0}{\nu\alpha_n}\cos\theta\cos(\lambda_n^0(d_0+l_0\tau\sin\theta))e^{i\alpha_n|L+l_0\tau\cos\theta|}, \quad \tau\in[-1,1]\end{aligned} \tag{3.3}$$

$$\tilde{\chi}(t) = \chi(x(t)), \quad \lambda_n^0 = \frac{\pi}{h}\left(n-\frac{1}{2}\right), \quad \alpha_n = \frac{\sqrt{k^2-\lambda_n^{02}}}{\sqrt{\nu}}, \quad n = 1, 2, \dots$$

The set $\{\alpha_n\}$ consists of a denumerable set of pure imaginary components and a finite number N of real components. The real α_n correspond to travelling waves in the layer, while the remainder characterize inhomogeneous modes, the amplitudes of which decrease exponentially.

Note that BIE (3.3) can be solved numerically by the boundary element method.[7,8] At the same time, for short cracks, to construct an approximate solution of BIE (3.3) it is possible to use an asymptotic approach based on a three-step asymptotic model.[9] At the first stage, a representation of a reference stress field on the crack is constructed. Then, the infinite sum on the right-hand side of BIE (3.3) can, with an accuracy $O(e^{-\gamma L})$, $\gamma > 0$, be replaced by a finite sum, retaining the first N terms.

For a rectilinear crack, with a low value of l_0, the next stage of constructing the wave field can be simplified considerably by determining the principal term of the asymptotic form of the function of the opening of the crack. For this, we investigate the asymptotic form of the kernels in the first term of BIE (3.3), assuming the linear dimension of the defect to be small. Letting $l_0 \to 0$, we obtain in the limit an integral equation with a constant right-hand side

$$\begin{aligned}&\int_{-1}^{1}\frac{\chi_*(t)}{(t-\tau)^2}dt = q_0, \quad \tau\in[-1,1]\\&\chi_*(t) = l_0^{-1}\tilde{\chi}(t), \quad q_0 = \frac{\tilde{F}^*(\theta,d_0,L)}{\sqrt{\nu}}, \quad \tilde{F}^*(\theta,d_0,L) = \frac{p_0}{L}F_1(\theta,d_0,L)\\&F_1(L,d_0,\theta) = \sum_{n=1}^{N}(-1)^{n+1}[\gamma_{1n}(d_0,\theta)+i\gamma_{2n}(d_0,\theta)]e^{i\alpha_n L}\\&\gamma_{1n}(d_0,\theta) = -\sin\theta\sin(\lambda_n^0 d_0), \quad \gamma_{2n}(d_0,\theta) = -\frac{\lambda_n^0}{\nu\alpha_n}\cos\theta\cos(\lambda_n^0 d_0)\end{aligned} \tag{3.4}$$

Integral Eq. (3.4) has the following solution in the class of bounded functions[11,12]

$$\chi_*(t) = \sqrt{1-t^2}W_0, \quad W_0 = -q_0/\pi \tag{3.5}$$

After determining the function of the opening of the crack, it is possible to construct the wave field in the layer, in particular on its surface. Evaluating the contour integral in formula (2.1) using the theory of residues, and singling out the amplitudes of the displacement field on the upper boundary ($x_1 > 0$) in the far zone, we obtain the following convenient formula for calculating the wave field

$$u(x_1, h) = u^s(x_1, h) + \sum_{n=1}^{N} A_n e^{i\alpha_n x_1} + O(e^{-\varepsilon x_1}), \quad \varepsilon > 0$$
$$A_n = (-1)^{n+1} l_0^2 P(L, d_0, \theta)[\gamma_{1n}(d_0, \theta) - i\gamma_{2n}(d_0, \theta)], \quad P(L, d_0, \theta) = \pi W_0/(2h) \tag{3.6}$$

which can then be used effectively to solve the inverse problem.

4. Identification of the parameters of a rectilinear crack

The solution of the inverse problem of reconstructing the crack parameters is based on the solution of the direct problem and additional information (1.4). For a rectilinear crack allowing of parameterization (3.2), the problem of identifying the crack reduces to the problem of determining the crack parameters l_0, d_0, θ and L. Note that the amplitudes of the surface waves are proportional to the square of the crack length.

Problem 1. Let us assume that the amplitude values $A_n^*(n = 1, 2)$ of the displacement field in the far zone of the upper boundary are specified as additional information. The identification procedure is carried out by frequency probing. For unique identification it is sufficient to examine two frequencies, k_1 and k_2, at each of which there are two travelling waves, $A_1^*(k_1)$ and $A_2^*(k_1)$ (the amplitudes of the first and second waves at frequency k_1), and $A_1^*(k_2)$ and $A_2^*(k_2)$ (the amplitudes of the first and second waves at frequency k_2), are specified. Using the expressions for the amplitudes (3.6), it is possible to reduce the problem of identifying a rectilinear crack to a stage-by-stage determination of the crack parameters, solving transcendental equations.

Stage 1. Determination of the depth of the crack d_0. To determine d_0, let us examine the ratio of the amplitudes of the first and second waves, which we will denote as

$$\mu_1 + i\mu_2 = -\frac{A_1^*(k_1)}{A_2^*(k_1)}$$

Then, taking into account the expression for the amplitudes (3.6), we obtain a homogeneous system of two equations in the parameters d_0 and θ that is linear in $\sin\theta$ and $\cos\theta$. From the solution of the system we obtain

$$d_0 = \frac{h}{\pi}\arccos\frac{-b \pm \sqrt{b^2 - 4a(c-a)}}{4a}$$
$$a = l_1(\mu_1^2 + \mu_2^2),\ b = -\mu_1(l_1 + l_2),\ c = l_2 - l_2\mu_1 + l_1\mu_1,\ l_1 = \frac{3\pi}{2h\alpha_2(k_1)},\ l_2 = \frac{\pi}{2h\alpha_1(k_1)} \tag{4.1}$$

Thus, for the unique determination of d_0, the values of the amplitudes at a single frequency are insufficient. For this it is necessary to know the amplitudes at a second frequency k_2, to carry out actions similar to analysis of the ratio of amplitudes at the first frequency and to obtain two roots of d_0, one of which will be the true value of d_0 and one of the solutions (4.1).

Stage 2. Determination of the angle of inclination of the crack θ, $0 \le \theta < \pi$. Using the value of d_0 found by means of formula (4.1), the angle of inclination of the crack can be determined from the formula

$$\theta = \pi - \operatorname{arcctg}\frac{c_{11}(d)}{c_{12}(d)} \tag{4.2}$$

where, if $c_{12}(d) = 0$, then $\theta = 0$.

Stage 3. Determination of the distance L from the point of application of the load to the vertical axis passing through the middle of the crack. The quantity L occurs in the expression for W_0, and, to determine this parameter, it is necessary to make two measurements of the amplitudes with different positions of the source. In the first case, the load is applied at a distance L_1 from the crack axis; the corresponding amplitudes are $A_1^*(k, L_1)$ and $A_2^*(k, L_1)$. In the second case, the load is applied at a distance $L_2 = L_1 - L_0$ from the crack axis; the corresponding amplitudes are $A_1^*(k, L_2)$ and $A_2^*(k, L_2)$. Then, to determine L_1, from Eq. (3.4) we have the relation

$$\frac{A_1^*(k, L_1)}{A_1^*(k, L_2)} = \frac{F_1(L_1, d_0, \theta)}{F_1(L_2, d_0, \theta)}$$

from which we obtain

$$L_1 = -\frac{\varphi + 2\pi m}{\alpha_1 - \alpha_2}, \quad m = 0, 1, \ldots, \quad \varphi = \arg\frac{a_2}{a_1}$$
$$a_j = (\gamma_{1j} + i\gamma_{2j})(A_1^*(k, L_1)e^{i\alpha_j L_0} - A_1^*(k, L_2)), \quad j = 1, 2 \tag{4.3}$$

Hence, the distance L_1 cannot be determined uniquely; knowing the amplitudes at a single frequency, it is possible to obtain only a certain collection of points on the Ox axis. For the unique determination of L_1 it is necessary to know the amplitudes at another frequency. Then, from the intersection of the set of points obtained at the first and second frequencies, we can determine the coordinate of L.

Numerical experiments showed that the quantity L_1 is determined with an error of less than 1% for accurate input data and is stable to noisy input information.

Stage 4. Determination of the crack length l_0. The quantity l_0 is determined from one of the expressions for the amplitudes, for example,

$$A_1 = l_0^2 P(L, d_0, \theta)[\gamma_{11}(d_0, \theta) - i\gamma_{21}(d_0, \theta)]$$

It must be pointed out that the parameter l_0 is determined with the least accuracy compared with the remaining parameters, as the error of its determination, besides the error of the input information, is affected by the error in identifying the parameters determined at the previous stages.

Calculations were carried out for a layer of thickness $h = 1$ of austenitic steel ($\nu = 0.64$). The frequencies $k_1 = 5$ and $k_2 = 6$ were selected, at each of which there were two propagating waves; $L_0 = -0.4$. The parameters d_0, θ and L_1 were determined from formulae (4.1)–(4.3). As expected, as the crack length increases, the corresponding relative errors in determining the crack parameters are $\varepsilon_{d_0} = |d_0 - d_0^a|/d_0^a$, $\varepsilon_\theta = |\theta - \theta^a|/\theta^a$ and $\varepsilon_{L_1} = |L_1 - L_1^a|/L_1^a$. The true values are $d_0^a = 0.5$, $\theta^a = \pi/3$ and $L_1^a = 5.6$.

From the numerical results on determining d_0, θ and L_1 as a function of the crack length l_0, presented in Fig. 1, it can be seen that, when $l_0 \leq 0.2h$, the error in determining the crack parameters d_0, θ and L_1 is less than 5% for accurate input data; the crack length is determined with an error of less than 10%. The results indicate that the procedure for identifying the crack parameters is fairly stable.

Fig. 2 shows graphs of the relative error in determining the parameters $d_0 = 0.5$, $\theta = \pi/4$ and $L_1 = 5.6$ as a function of the degree of noisyness of the input data, η, at frequencies $k_1 = 5$ and $k_2 = 7.8$. It can be seen that, with an error in prescribing the input data $\eta \approx 20\%$, the parameters d_0, θ and L_1 are determined with an error of about 1% for $l_0 = 0.01$

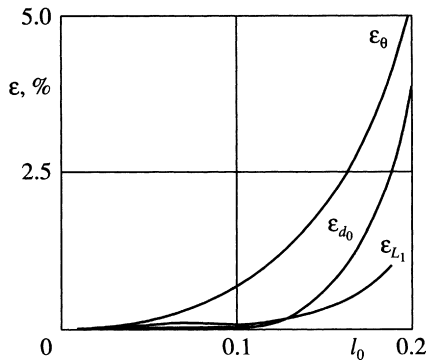

Fig. 1.

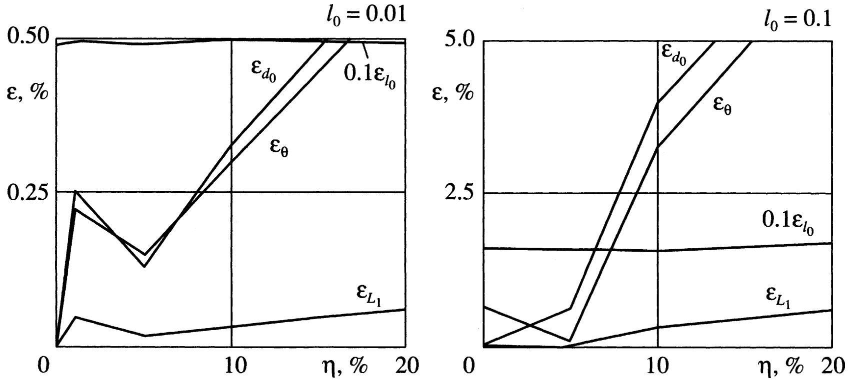

Fig. 2.

and with an error of about 5% for $l_0 = 0.1$. As the crack length increases, the error in determining the parameters d_0, θ and L_1 depends to a greater extent on the accuracy of the input data.

Problem 2. The kernel of BIE (2.4), as in Problem 1, can be represented in the form of the sum of irregular and regular parts

$$K_{ji}(x, y) = K_{ji}^{(0)}(x, y) + K_{ji}^{(1)}(x, y)$$
$$K_{ji}^{(s)}(x, y) = \frac{1}{2\pi}\int_{\sigma} k_{ji}^{(s)}(\alpha_1, x, y) e^{i\alpha_1(x_1 - y_1)} d\alpha_1, \quad s = 0, 1 \tag{4.4}$$

The structure of the kernel (4.4) was investigated, the main terms of the asymptotic forms of the integrands $k_{ji}^{(s)}(\alpha_1, x, y)$ as $|\alpha_1| \to \infty$ were singled out and it was found that $k_{ji}^{(1)}(\alpha_1, x, y)$ are functions decreasing at infinity, while $k_{ji}^{0}(\alpha_1, x, y)$ are functions increasing at $x_3 = y_3$, and here the corresponding integrals are understood in the sense of the final Hadamard value.[11] For cracks allowing of the previously introduced parameterization $x_j = q_j(t)$ and $y_j = q_j(\tau)$, BIE (2.4) can be reduced to the form

$$\int_{-1}^{1}\left[\frac{R_{ji}(t, \tau)}{(t-\tau)^2} + \tilde{K}_{ji}^{(1)}(t, \tau)\right]\tilde{\chi}_j(t) g(t) dt = \tilde{F}_i(\tau), \quad \tau \in [-1, 1]$$

$$R_{ji}(t, \tau) = \sum_{m=1}^{2}\left[2\frac{\Delta_m^-(\tau)}{(\Delta_m^+(\tau))^2} M_{ji}^{(1)}(\nu_m, q'(t), q'(\tau)) - 4\nu_m \frac{q_1'(\tau) q_3'^2(\tau)}{(\Delta_m^+(\tau))^2} M_{ji}^{(2)}(\nu_m, q'(t), q'(\tau))\right]$$

$$\Delta_m^+(\tau) \neq 0, \quad \tau \in [-1, 1] \tag{4.5}$$

$$\Delta_m^{\pm}(\tau) = \nu_2^2 q_3'^2(\tau) \pm q_1'^2(\tau)$$

$$\nu_{1,2} = \sqrt{(\gamma_1 - 2\gamma_7\gamma_5 - \gamma_7^2 \mp w)/(2\gamma_5)}, \quad w = \sqrt{(\gamma_7^2 - \gamma_1)(\gamma_7^2 - \gamma_1 + 4\gamma_5(\gamma_5 + \gamma_7))}$$

$$\tilde{\chi}_j(t) = \chi_j(x(t)), \quad \tilde{K}_{ji}^{(1)}(t, \tau) = K_{ji}^{(1)}(x(t), y(\tau)), \quad \tilde{F}_i(\tau) = F_i(y(\tau))$$

where $M_{ji}^{(1)}$ and $M_{ji}^{(2)}$ are continuous functions, which depend on the constants of the material and on the components of the vector of the normal at points of the curve l.

For a crack allowing of parameterization of the form (3.2), BIE (4.5) becomes

$$l_0^{-1}\int_{-1}^{1}\left[\frac{G_{ji}(\theta)}{(t-\tau)^2}+l_0^2\tilde{K}_{ji}^{(1)}(t,\tau)\right]\tilde{\chi}_j(t)dt = \tilde{F}_i(\tau),\quad \tau\in[-1,1]$$

$$G_{ji}(\theta) = \sum_{m=1}^{2}\left[2\frac{\tilde{\Delta}_m^-(\theta)}{(\tilde{\Delta}_m^+(\theta))^2}\tilde{M}_{ji}^{(1)}(\nu_m,\theta)-4\frac{\nu_m\sin\theta\cos\theta}{(\tilde{\Delta}_m^+(\theta))^2}\tilde{M}_{ji}^{(2)}(\nu_m,\theta)\right] \tag{4.6}$$

$$\tilde{\Delta}_m^{\pm}(\theta) = \nu_m^2\sin^2\theta\pm\cos^2\theta,\quad \tilde{M}_{ji}^{(k)}(\nu_m,\theta) = M_{ji}^{(k)}(\nu_m,q'(t),q'(\tau))$$

Note that, when the rectilinear crack is vertical ($\theta = \pi/2$) or horizontal ($\theta = 0$), BIE system (4.6) breaks down into two independent BIEs: one in $\tilde{\chi}_1$ and the other in $\tilde{\chi}_3$; in the case of an arbitrary rectilinear crack, this does not occur, and the components of the jumps are interconnected.

Below, considering cracks of small relative length, as above, in the limit as $l_0 \to 0$, we obtain a system of integral equations with constant right-hand sides

$$\int_{-1}^{1}\frac{G_{ji}(\theta)}{(t-\tau)^2}\tilde{\chi}_j^*(t)dt = \tilde{F}_i^*(L,d_0,\theta),\quad \tau\in[-1,1]$$
$$\tilde{\chi}_j^*(t) = l_0^{-1}\tilde{\chi}_j(t),\quad \tilde{F}_i^*(L,d_0,\theta) = \lim_{l_0\to 0}\tilde{F}_i(\tau) \tag{4.7}$$

This system has a solution in the class of bounded functions[11,12] of the form

$$\tilde{\chi}_j(t) = l_0\sqrt{1-t^2}W_{0j},\quad j = 1,3$$

$$W_{01}(L,d_0,\theta) = \frac{\tilde{F}_3^*(L,d_0,\theta)G_{13}(\theta)-\tilde{F}_1^*(L,d_0,\theta)G_{33}(\theta)}{G(\theta)}\ (1\leftrightarrow 3) \tag{4.8}$$

$$G(\theta) = G_{11}(\theta)G_{33}(\theta)-G_{13}(\theta)G_{31}(\theta)$$

Then, the components of the displacement fields at the upper boundary can be represented in the form

$$u_j(x_1,h) = u_j^s(x_1,h)+\sum_{n=1}^{N}A_{jn}e^{i\beta_n x_1}+O(e^{-\varepsilon x_1}),\quad j = 1,3$$
$$A_{jn} = l_0^2[A_{jn}^{(1)}(d_0,\theta)W_{01}+A_{jn}^{(3)}(d_0,\theta)W_{03}] \tag{4.9}$$

and here, as in the case of Problem 1, the wave amplitudes are proportional to the square of the crack length.

The identification procedure can be carried out in terms of the values of the amplitudes of the wave fields or the components of the displacements on the upper boundary of the layer. The statement of the inverse problem, in which the wave fields on the upper boundary of the layer are themselves specified as additional information, was considered. In this case, the uniqueness of the solution of the inverse problem depends on the position of the probing points, i.e. the points at which the wave fields of displacements and the number of travelling waves are measured. Numerical analysis of the inverse problem showed that, for the unique determination of the crack parameters during positional probing, it is sufficient to measure the wave fields of displacements at two points at a frequency at which there are two travelling waves.

Fig. 3 presents the results of numerical experiments to determine some of the parameters of the rectilinear crack for a layer of austenitic steel; in the layer there are four propagating modes at $k = 4.9$. Graphs of the relative error of determining the parameters d_0 and θ as a function of crack length are given. The true values of the parameters are $d_0^a = 0.33$ and $\theta^a = 65°$.

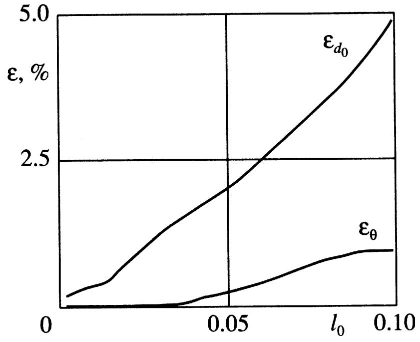

Fig. 3.

The proposed asymptotic approach enables one to determine the parameters of a rectilinear crack whose length is no more than 20% of the layer thickness, with an error of less than 1% in the case of accurate input data, which indicates the efficiency of the model for calculating wave fields based on the asymptotic approach, and provides a fairly stable procedure for identifying short rectilinear cracks.

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